# Minimal and maximal solutions of two point boundary value problems for the equation $f(t,x,x',x'')=0$

by

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**Abstract:** In this article we consider a boundary value problem for the equation $f(t,x,x',x'')=0$ with mixed boundary conditions. For this problem, assuming the existence of suitable barrier strips and using the monotone iterative method, we derive the minimal and maximal solutions.

**Keywords:** Boundary value problems, existence, minimal and maximal solutions, monotone method, barrier strips.

**2000 Mathematical Subject Classification:** 34B15.

**1. Introduction.** In this paper we apply the monotone iterative method to obtain minimal and maximal solutions to the nonlinear boundary value problem (BVP)

$$\begin{cases} f(t,x,x',x'')=0, \quad 0\le a\le t\le b, \\ x(a)=A, \quad x'(b)=B, \end{cases} \tag{1.1}$$

where the scalar function $f(t,x,p,q)$ is continuous and has continuous first derivatives only on suitable subsets of $[a,b]\times R^3$.

For results, which guarantee the existence of $C^2[a,b]$-solutions to BVPs for the equation $x''=f(t,x,x',x'')-y(t)$ with various linear boundary conditions, see [6,7,17,18,21-23]. Conserning the uniqueness results, we refer to [21]. A result, conserning the existence and uniqueness of $C^2[a,b]$-solutions to the BVP for the equation $x''=f(t,x,x',x'')$, with general linear boundary conditions, can be found in [27].The results of [19] guarantee the existence of $W^{2,\infty}[a,b]$-solutions or of $C^2[a,b]$-solutions to the Dirichlet BVP for the equation $f(t,x,x',x'')=0$, where the function $f(t,x,p,q)$ is defined on $[a,b]\times R^n\times R^n\times Y$, and $Y$ is a non-empty closed connected or locally connected subset of $R^n$. Finally, the $C^2[a,b]$-solvability of BVPs for the equation $f(t,x,x',x'')=0$ with fully nonlinear boundary conditions is studied in [12].

Note that, in the literature, the monotone iterative method is applied on BVPs for equations of the forms $x''=f(t,x,x')$ and $(\phi(x'))'=f(t,x,x')$ with various boundary conditions (see, for example, [2-5,9-11,13,15,20,26,28]). The sequences of iterates, considered in [2-5,10,13,28], converge to the extremal solutions, while the sequences of iterates, considered in [9,15,19], converge to the unique solution. The first elements $u_0(t)$ and $v_0(t)$ of such sequences of iterates usually are lower and upper solutions respectively of the problems under consideration (see, for example, [2-5,10,13,28]). In order to derive the needed monotone iterates, the authors of [2-5,10,13,15,28] use suitable growth conditions. For more applications of the monotone iterative method, see [1,14,16,25,29].

In this article, following [11], we obtain the extremal solutions to (1.1) under assumption of the existence of suitable barrier strips (see Remarks 2.1 and 2.2 below), which immediately imply the first iterates $u_0(t)$ and $v_0(t)$. A version of Theorem 5.1 of [12] implies the existence of the next iterates, and a suitable comparison result guarantees the monotone properties for the sequences of iterates. Finally, the Arzela-Askoli's theorem ensures the existence of the extremal solutions of the problem (1.1) as limits of the sequences of iterates.

In order to obtain our results, in what follows, we will make the following basic hypotheses.

**2. Basic hypotheses.** The following four hypotheses will be a tool to obtain our results.

**H1.** There are a constant $K > 0$ and constants $F, F_1, L, L_1$ such that

$$Fa \le A \le La, \quad F_1 < F \le B \le L < L_1$$

and for $T := \{(t,x) : a \le t \le b, Ft \le x \le Lt\}$

$$f(t,x,p,q) + Kq \ge 0 \text{ on } \Big\{(t,x,p,q) : (t,x) \in T,\ p \in [L, L_1],\ q \in (-\infty, 0)\Big\},$$

$$f(t,x,p,q) + Kq \le 0 \text{ on } \Big\{(t,x,p,q) : (t,x) \in T,\ p \in [F_1, F],\ q \in (0, \infty)\Big\}.$$

**Remark 2.1.** Set $\Phi_1(t,x,p,q) \equiv f(t,x,p,q) + Kq$. Then, the strip $\Delta_1 = [a,b] \times [L, L_1]$, on which $\Phi_1(t,x,p,q) \ge 0$, and the strip $\Delta_2 = [a,b] \times [F_1, F]$, on which $\Phi_1(t,x,p,q) \le 0$, are such that the graph of the function $x'(t)$, $t \in [a,b]$, does not cross $\Delta_1$ and $\Delta_2$, and is located between them. For this reason $\Delta_1$and $\Delta_2$ are called *barrier strips* for $x'(t)$, $t \in [a,b]$.

**H2.** There are constants $G_i^-$, $G_i^+$, $H_i^-$, $H_i^+$, $i = 1,2$, such that

$$G_2^+ > G_1^+ \ge 2C, \quad G_2^- > G_1^- \ge 2C, \quad H_2^+ < H_1^+ \le -2C, \quad H_2^- < H_1^- \le -2C,$$

where $C = \max\{|L|, |F|\}/(b-a)$,

$$\begin{cases} f(t,x,p,q) \text{ and } f_q(t,x,p,q) \text{ are continuous and } f_q(t,x,p,q) < 0 \text{ for} \\ (t,x,p,q) \in [a,b] \times [m_1 - \varepsilon, M_1 + \varepsilon] \times [F - \varepsilon, L + \varepsilon] \times [m_2 - \varepsilon, M_2 + \varepsilon], \\ \text{where: } m_1 = \min\{Fa, Fb\},\ M_1 = \max\{La, Lb\},\ m_2 = \min\Big\{H_2^+, H_2^-\Big\}, \\ M_2 = \max\Big\{G_2^+, G_2^-\Big\}, \text{ and } \varepsilon > 0 \text{ is fixed and such that } H_1^+ > H_2^+ + \varepsilon, \\ H_1^- > H_2^- + \varepsilon,\ G_2^+ > G_1^+ + \varepsilon,\ G_2^- > G_1^- + \varepsilon; \end{cases} \tag{2.1}$$

$f_t(t,x,p,q)$, $f_x(t,x,p,q)$ and $f_p(t,x,p,q)$ are continuous for

$$(t,x,p,q) \in [a,b] \times [m_1, M_1] \times [F, L] \times [m_2, M_2],$$

$f_t(t,x,p,q) + f_x(t,x,p,q)p + f_p(t,x,p,q)q \ge 0$ for

$$(t,x,p,q) \in [a,b] \times [m_1, M_1] \times [F, L] \times \Big([H_2^+, H_1^+] \cup [G_1^+, G_2^+]\Big),$$

and $f_t(t,x,p,q) + f_x(t,x,p,q)p + f_p(t,x,p,q)q \le 0$ for

$$(t,x,p,q) \in [a,b] \times [m_1, M_1] \times [F, L] \times \Big([H_2^-, H_1^-] \cup [G_1^-, G_2^-]\Big),$$

where $F$ and $L$ are the constants of **H1**.

**Remark 2.2.** Set $\Phi_2(t,x,p,q) \equiv f_t(t,x,p,q) + f_x(t,x,p,q)p + f_p(t,x,p,q)q$. Then, the pair of strips $\Omega_1 = [a,b] \times ([H_2^+, H_1^+] \cup [G_1^+, G_2^+])$, on which $\Phi_2(t,x,p,q) \geq 0$, and the pair of strips $\Omega_2 = [a,b] \times ([H_2^-, H_1^-] \cup [G_1^-, G_2^-])$, on which $\Phi_2(t,x,p,q) \leq 0$, are such that the graph of the function $x''(t)$, $t \in [a,b]$, can not cross the outer strips, of the four such ones, defined by $\Omega_1$and $\Omega_2$. For this reason the outer strips of $\Omega_1$and $\Omega_2$ are called *barrier strips* for $x''(t)$, $t \in [a,b]$.

**H3.** For $m_3 = \min\{H_1^+, H_1^-\}$ and $M_3 = \max\{G_1^+, G_1^-\}$

$$h(\lambda,t,x,p,m_3-\varepsilon)h(\lambda,t,x,p,M_3+\varepsilon) \leq 0 \text{ for } (\lambda,t,x,p) \in [0,1]\times[a,b]\times[m_1-\varepsilon, M_1+\varepsilon]\times[F-\varepsilon, L+\varepsilon]$$

where $h(\lambda,t,x,p,q) = (\lambda-1)Kq + \lambda f(t,x,p,q)$, $F$, $L$ and $K$ are the constants of **H1**, and $H_1^+$, $H_1^-$, $G_1^+$, $G_1^-$, $C$, $m_1$, $M_1$ and $\varepsilon$ are as in **H2**.

**H4.** $f_x(t,x,p,q) \geq 0$ for $(t,x,p,q) \in T \times [F,L] \times \left[\min\left\{H_1^+, H_1^-\right\}, \max\left\{G_1^+, G_1^-\right\}\right]$, where the trapezoid $T$ and the constants $F$ and $L$ are as in **H1**, and $H_1^+$, $H_1^-$, $G_1^+$ and $G_1^-$ are the constans of **H2**, and $m_3$ and $M_3$ are as in **H3**.

**3. The main result.** For any function $y(t) \in C[a,b]$ bounded on $[a,b]$, we define a map $\mathcal{A}$ as follows

$$x = \mathcal{A}y,$$

if and only if $x(t) \in C^2[a,b]$ is a solution to the BVP

$$\begin{cases} f\Big(t, y(t), x', x''\Big) = 0, & t \in [a,b], \\ x(a) = A, \ x'(b) = B. \end{cases} \tag{3.1}$$

We will show that under the hypotheses **H1** , **H2** and **H3** the map $\mathcal{A}$ is uniquely determined. For this reason, we consider two sequences $\{u_n\}$ and $\{v_n\}$, $n = 0, 1, ...$, defined by the formulas

$$u_{n+1} = \mathcal{A}u_n \quad \text{and} \quad v_{n+1} = \mathcal{A}v_n,$$

where $u_0 = Ft$, $v_0 = Lt$, $t \in [a,b]$, and $F$ and $L$ are as in **H1**. Now we formulate the following our main result.

**Theorem 3.1.** *Let the hypotheses* **H1 - H4** *be hold. Then there are sequences* $\{u_n\}$ *and* $\{v_n\}, n = 0, 1, ...,$ *such that for* $n \to +\infty$

$$u_n \to u^m, \quad v_n \to v^M \quad \textit{and} \quad u_0 \leq u_1 \leq ... \leq u_n \leq ... \leq u^m \leq x \leq v^M \leq ... \leq v_n \leq ... \leq v_1 \leq v_0,$$

*where* $u^m(t)$ *and* $v^M(t)$ *are the minimal and maximal solutions of the BVP (1.1) respectively, and* $x(t) \in C^2[a,b]$ *is a solution of (1.1).*

The proof of this statement can be found at the end of this article and is based on the auxiliary results, which we present in the next section.

**4. Auxiliary statements.** We begin this section with an existence result, which is a modification of Theorem 6.1 of [8, Chapter II]. Namely, we consider the family of BVPs

$$\begin{cases} Kx'' = \lambda\Big(Kx'' + f(t, y(t), x', x'')\Big), & t \in [a,b], \\ x(a) = A, \ x'(b) = B, \end{cases} \tag{4.1$_\lambda$}$$

where $\lambda \in [0,1]$ and $K > 0$, and formulate the following

**Lemma 4.1.** *Assume that there are constants* $Q_i$, $i = 0, 1, ..., 5$, *independent of* $\lambda$ *such that:*

**(i)** *For each solution* $x(t) \in C^2[a,b]$ *of* $(4.1)_\lambda$ *it holds*

$$Q_0 < x(t) < Q_1,\ Q_2 < x'(t) < Q_3,\ Q_4 < x''(t) < Q_5,\ t \in [a,b].$$

*Moreover, assume that:*

**(ii)** $f(t,x,p,q)$ *and* $f_q(t,x,p,q)$ *are continuous, and* $f_q(t,x,p,q) < 0$ *for* $(t,x,p,q) \in [a,b] \times [Q_0,Q_1] \times [Q_2,Q_3] \times [Q_4,Q_5]$,

**(iii)** $h(\lambda,t,x,p,Q_4)h(\lambda,t,x,p,Q_5) \le 0$ *for* $(\lambda,t,x,p) \in \Lambda := [0,1] \times [a,b] \times [Q_0,Q_1] \times [Q_2,Q_3]$, *where* $h(\lambda,t,x,p,q) = (\lambda-1)Kq + \lambda f(t,x,p,q)$.

*Then the BVP (3.1) has a* $C^2[a,b]$*-solution for each* $y(t) \in C[a,b]$ *such taht* $Q_0 < y(t) < Q_1$, $t \in [a,b]$.

*Proof.* In view of **(ii)** and **(iii)**, we conclude that there is a unique function $G(\lambda,t,x,p)$ which is continuous on $\Lambda$ and such that

$$q = G(\lambda,t,x,p) \text{ for } (\lambda,t,x,p) \in \Lambda$$

is equivalent to the equation

$$h(\lambda,t,x,p,q) = 0 \quad \text{on} \quad \Lambda \times [Q_4,Q_5].$$

Note that $h(0,t,x,p,0) = 0$ yields

$$G(0,t,x,p) = 0 \text{ for } (t,x,p) \in [a,b] \times [Q_0,Q_1] \times [Q_2,Q_3]. \tag{4.2}$$

Thus, the family $(4.1)_\lambda$ is equivalent to the family of BVPs

$$\begin{cases} x'' = G(\lambda,t,y(t),x'), & t \in [a,b], \\ x(a) = A, \ x'(b) = B, \end{cases} \tag{4.3}$$

where $\lambda \in [0,1]$.

Now, define the set

$$U = \Big\{x(t) \in C^2[a,b] : x(t) \in (Q_0,Q_1),\ x'(t) \in (Q_2,Q_3),\ x''(t) \in (Q_4,Q_5)\Big\},$$

which is an open subset of the convex set $C^2_Q[a,b]$ of the Banach space $C^2[a,b]$ and consider the map

$$\mathbf{N} : C^2_Q[a,b] \to C[a,b], \quad \text{defined by} \quad \mathbf{N}x = x'',$$

where $C^2_Q[a,b] = \{x \in C^2[a,b] : x(a) = A,\ x'(b) = B\}$. It is easy to see that the map

$$\mathbf{S} : C^2_{Q_0}[a,b] \to C[a,b], \text{ where } \mathbf{S}x = x'' \text{ and } C^2_{Q_0}[a,b] = \{x \in C^2[a,b] : x(a) = 0,\ x'(b) = 0\},$$

is one-to-one and the problem $\mathbf{S}x = 0$, $x(a) = A$, $x'(b) = B$, has a unique solution $l$. Then $\mathbf{N}^{-1} : C[a,b] \to C^2_Q[a,b]$ exists, is continuous, and moreover

$$\mathbf{N}^{-1}s = \mathbf{S}^{-1}s + l.$$

Let $\mathbf{H}_\lambda : \overline{U} \to C_Q^2[a,b]$ be defined by

$$\mathbf{H}_\lambda x = \mathbf{N}^{-1}\mathbf{G}_\lambda \mathbf{j}(x),\ \lambda \in [0,1],$$

where

$$\mathbf{j}:\ C_Q^2[a,b] \to C^1[a,b] \quad \text{is defined by} \quad \mathbf{j}x = x,$$

$$\mathbf{G}_\lambda : C^1[a,b] \to C[a,b] \quad \text{is defined by} \quad (\mathbf{G}_\lambda x)(t) = G\Big(\lambda, t, y(t), x'(t)\Big),\ \lambda \in [0,1].$$

Clearly, $\mathbf{H}_\lambda$ is a compact homotopy, because $\mathbf{j}$ is a completely continuous embeding, and $\mathbf{G}_\lambda$ and $\mathbf{N}^{-1}$ are continuous. Moreover, $\mathbf{H}_\lambda x = x$ implies

$$x = \mathbf{N}^{-1}\mathbf{G}_\lambda \mathbf{j}(x).$$

Hence, by the definition of $\mathbf{N}^{-1}$, we have

$$x = \mathbf{S}^{-1}\mathbf{G}_\lambda \mathbf{j}(x) + l.$$

Finally, since $\mathbf{S}l = 0$, it follows that

$$\mathbf{S}x = \mathbf{G}_\lambda \mathbf{j}(x).$$

Thus, the fixed points of $\mathbf{H}_\lambda$ are solutions to (4.3) and obviously $\mathbf{H}_\lambda$ has no fixed points on $\partial U$. In view of (4.2), the map $\mathbf{H}_0$, which has the form $\mathbf{H}_0 x = l$, is constant. Moreover, $l$, as the unique solution of $(4.1)_0$, belongs to the set $U$. Hence, by Theorem 2.2 of [8], the map $\mathbf{H}_0$ is essential. The topological transversality theorem of [8] implies that $\mathbf{H}_1$ is also essential, i.e. for $\lambda = 1$ (4.3) has a solution. Moreover, for $\lambda = 1$ (4.3) coinsides with (3.1). Therefore, the problem (3.1) has a solution. The proof of the lemma is complete.□

In order to obtain our next auxiliary results, we introduce the following two sets

$$V = \{y(t) \in C[a,b] :\ Ft \le y(t) \le Lt,\ t \in [a,b]\},$$

$$V_1 = \{y(t) \in C^1[a,b] :\ Ft \le y(t) \le Lt,\ F \le y'(t) \le L,\ t \in [a,b]\},$$

where the constants $L$ and $F$ are as in H1, and formulate the following results.

**Lemma 4.2.** *Let* **H1** *be hold and* $x(t) \in C^2[a,b]$ *be a solution to* $(4.1)_\lambda$ *with* $y(t) \in V$. *Then the following statements hold:*

*(i) If there is an interval* $T_1 \subseteq [a,b]$ *such that*

$$L \le x'(t) \le L_1 \quad \textit{for} \quad t \in T_1, \tag{4.4}$$

*then* $x''(t) \ge 0$ *for* $t \in T_1$.

*(ii) If there is an interval* $T_2 \subseteq [a,b]$ *such that* $F_1 \le x'(t) \le F$ *for* $t \in T_2$, *then* $x''(t) \le 0$ *for* $t \in T_2$.

*Proof.* Since the proofs of (i) and (ii) are similar, it is enough to show that (4.4) implies $x''(t) \ge 0$ for $t \in T_1$. Indeed, the assertion is true for $\lambda = 0$. Now, let $\lambda \in (0,1]$ and assume that there is a $t_0 \in T_1$ such that $x''(t_0) < 0$. Then

$$0 > Kx''(t_0) = \lambda\left[Kx''(t_0) + f(t_0, x(t_0), x'(t_0), x''(t_0))\right] \ge 0.$$

The obtained contradiction proves the assertion. □

**Lemma 4.3.** *Let* **H1** *be hold, and* $x(t) \in C^2[a,b]$ *be a solution to (4.1)*$_\lambda$ *with* $y(t) \in V$. *Then*

$$Ft \le x(t) \le Lt, \quad F \le x'(t) \le L \quad \textit{for} \quad t \in [a,b].$$

*Proof.* Consider the sets

$$Y_0 = \{t \in [a,b] : L < x'(t) \le L_1\} \text{ and } Y_1 = \{t \in [a,b] : F_1 \le x'(t) < F\}$$

and suppose that they are not empty. Then, using the continuity of $x'(t)$ and the inequality $F \le x'(b) \le L$, we easily conclude that there are closed intervals

$$[t_0, \tau_0] \subseteq Y_0 \quad \text{and} \quad [t_1, \tau_1] \subseteq Y_1$$

such that

$$x'(t_0) > x'(\tau_0) \quad \text{and} \quad x'(t_1) < x'(\tau_1). \tag{4.5}$$

On the other hand, by Lemma 4.2, we have

$$x''(t) \ge 0 \text{ for } t \in [t_0, \tau_0] \quad \text{and} \quad x''(t) \le 0 \text{ for } t \in [t_1, \tau_1]$$

and therefore, we have

$$x'(t_0) \le x'(\tau_0) \quad \text{and} \quad x'(t_1) \ge x'(\tau_1).$$

But this contradicts (4.5).The obtained contradiction shows that $Y_0$ and $Y_1$ are empty, and so we see that

$$F \le x'(t) \le L \quad \text{for} \quad t \in [a,b].$$

Integrating this expresion from $a$ to $t$ and using the fact that $Fa \le A \le La$, we get

$$Ft \le x(t) \le Lt, \ \ t \in [a,b]$$

which concludes the proof.□

**Remark 4.1.** Let $x(t) \in C^2[a,b]$ be a solution to (1.1). Then, in view of Lemma 4.3, if $F = L$, it follows that $x'(t) = B$, $t \in [a,b]$. Now, using $Fa \le A \le La$, we see that $x(t) = Bt$, $t \in [a,b]$, is the unique $C^2[a,b]$-solution to the problem (1.1).

**Lemma 4.4.** *Let* **H1** *and* **H2** *be hold, and* $x(t) \in C^2[a,b]$ *be a solution to (4.1)*$_\lambda$ *with* $y(t) \in V_1$. *Then*

$$m_3 \le x''(t) \le M_3, \ t \in [a,b],$$

*and there is a constant* $D$ *independent of* $\lambda$ *such that*

$$|x'''(t)| \le D \ \textit{for} \ t \in [a,b].$$

*Proof.* By the mean value theorem, there is a $\xi \in (a,b)$ such that $x''(\xi) = [x'(b) - x'(a)]/(b-a)$. Since Lemma 4.3 implies

$$F \le x'(t) \le L \quad \text{for} \quad t \in [a,b], \tag{4.6}$$

we see that

$$x''(\xi) \le 2C \le G_1^+, \tag{4.7}$$

where $C = \max\{|L|, |F|\}/(b-a)$. Now suppose that the set

$$Y = \{t \in [a, \xi] : G_1^+ < x''(t) \leq G_2^+\}$$

is not empty. The continuity of $x''(t)$ and (4.7) imply that there is a closed interval

$$[t_0, \tau_0] \subseteq Y \quad \text{such that} \quad x''(t_0) > x''(\tau_0). \tag{4.8}$$

Since (4.6) holds for $t \in [t_0, \tau_0]$ and

$$G_1^+ < x''(t) \leq G_2^+ \text{ for } t \in [t_0, \tau_0],$$

$$\begin{cases} m_1 \leq Ft \leq y(t) \leq Lt \leq M_1 & \text{for } t \in [t_0, \tau_0], \\ F \leq y'(t) \leq L & \text{for } t \in [t_0, \tau_0], \end{cases} \tag{4.9}$$

in view of **H2,** we have

$$\Psi_1(t) \equiv f_q\Big(t, y(t), x'(t), x''(t)\Big) < 0, \ t \in [t_0, \tau_0],$$

and for $t \in [t_0, \tau_0]$

$$\Psi_2(t) \equiv f_t\Big(t, y(t), x'(t), x''(t)\Big) + f_x\Big(t, y(t), x'(t), x''(t)\Big)y'(t) + f_p\Big(t, y(t), x'(t), x''(t)\Big)x''(t) \geq 0.$$

Thus, using the last two inequalities and the continuity of $f_t$, $f_x$, $f_p$ and $f_q$ on $[t_0, \tau_0]$, we conclude that $x'''$ is continuous on $[t_0, \tau_0]$ and

$$x'''(t) = \lambda\Psi_2\ (t)/[K(1-\lambda) - \lambda\Psi_1(t)] \ \geq 0 \quad \text{for} \quad t \in [t_0, \tau_0]. \tag{4.10}$$

Consequently,

$$x''(t_0) \leq x''(\tau_0),$$

which contradicts (4.8). Thus,

$$x''(t) \leq G_1^+ \quad \text{for} \quad t \in [a, \xi].$$

The inequality

$$H_1^- \leq x''(t), \ \ t \in [a, \xi]$$

can be obtained in the same manner.

Similarly, it is easy to show that

$$H_1^+ \leq x''(t) \leq G_1^-, \ \ t \in [\xi, b].$$

Finally, using (4.6), (4.9), the fact that $x''$ is bounded on $[a, b]$ and the continuity of the partial derivatives of $f(t, x, p, q)$ on the set $[a, b] \times [m_1, M_1] \times [F, L] \times [m_3, M_3]$, from (4.10) it follows that there is a constan $D$ independent of $\lambda$ such that

$$|x'''(t)| \leq D \text{ for } t \in [a, b].$$

The proof of the lemma is complete. □

**Lemma 4.5.** *Suppose that* **H1**, **H2** *and* **H3** *are hold. Then the BVP (3.1) has a* $C^2[a, b]$*-solution, if* $y(t) \in V_1$.

*Proof.* Let $x(t) \in C^2[a, b]$ be a solution to $(4.1)_\lambda$. Then, by Lemma 4.3, we have

$$F - \varepsilon < x'(t) < L + \varepsilon \quad \text{for} \quad t \in [a, b] \quad \text{and}$$

$$m_1 - \varepsilon < x(t) < M_1 + \varepsilon \quad \text{for } t \in [a,b],$$

while, by Lemma 4.4, we see that

$$m_3 - \varepsilon < x''(t) < M_3 + \varepsilon \quad \text{for} \quad t \in [a,b],$$

where $\varepsilon > 0$ is as in **H2**. Thus, the condition **(i)** of Lemma 4.1 holds for $Q_0 = m_1 - \varepsilon$, $Q_1 = M_1 + \varepsilon$, $Q_2 = F - \varepsilon$, $Q_3 = L + \varepsilon$, $Q_4 = m_3 - \varepsilon$ and $Q_5 = M_3 + \varepsilon$. Moreover, from (2.1) and **H3** it follows that the conditions **(ii)** and **(iii)** of Lemma 4.1 are satisfied. Besides,

$$m_1 - \varepsilon < y(t) < M_1 + \varepsilon \quad \text{for} \quad t \in [a,b].$$

So, we can apply Lemma 4.1 to conclude that the problem (3.1) has a solution in $C^2[a,b]$. The proof of the lemma is complete.□

Below, we need the following two lemmas which are adopted from [24].

**Lemma 4.6.** *[24, Chapter I, Theorem 1] Suppose $\phi(t)$ satisfies the differential inequality*

$$\phi'' + g(t)\phi' \geq 0 \quad \textit{for} \quad a < t < b, \tag{4.11}$$

*with $g(t)$ a bounded function. If $\phi(t) \leq M$ in $(a,b)$ and if the maximum $M$ of $\phi$ is attained at an interior point $c$ of $(a,b)$, then $\phi \equiv M$.*

**Lemma 4.7.** *[24, Chapter I, Theorem 2] Suppose $\phi(t)$ is a nonconstant function which satisfies the inequality (4.11) and has one-sided derivatives at $a$ and $b$, and suppose $g$ is bounded on every closed subinterval of $(a,b)$. If the maximum of $\phi$ occurs at $t = a$ and $g$ is bounded below at $t = a$, then $\phi'(a) < 0$. If the maximum occurs at $t = b$ and $g$ is bounded above at $t = b$, then $\phi'(b) > 0$.*

**Lemma 4.8.** *Suppose that $\phi \in C^2(a,b) \cap C^1[a,b]$ satisfies the inequality*

$$\phi''(t) + g(t)\phi'(t) \geq 0 \quad \textit{for} \quad t \in (a,b),$$

*where $g(t)$ is bounded on $(a,b)$. If $\phi(a) \leq 0$ and*

$$\phi'(b) \leq 0, \tag{4.12}$$

*then*

$$\phi(t) \leq 0 \quad \textit{for} \quad t \in [a,b]. \tag{4.13}$$

*Proof.* First, assume that $\phi(t)$ achieves its maximum at $t_0 \in (a,b)$. By Lemma 4.6, for $t \in [a,b]$ we obtain $\phi(t) \equiv \phi(t_0) = \phi(a) \leq 0$ and so (4.13) holds.

Next, suppose that $\phi(t)$ achieves its maximum at the ends of the interval $[a,b]$. If we assume $\phi(t) \leq \phi(b)$, $t \in [a,b]$, the application of Lemma 4.7 shows that $\phi'(b) > 0$, which contradicts (4.12). Thus, by our assumtions, $\phi(t) \leq \phi(a) \leq 0$, $t \in [a,b]$, and so (4.13) follows. The proof is complete.□

In the last two lemmas we make use the map $\mathcal{A}$ defined in the section 3.

**Lemma 4.9.** *Let* **H1**, **H2** *and* **H3** *be hold. Then, for any $y \in V_1$, the image $x$ by the map $\mathcal{A}$ exists and it is unique.*

*Proof.* The existence of the image of $x$ follows from Lemma 4.5. In order to see that $x$ is unique, fix $y$ and assume that $z$ is an other image of $y$ by $\mathcal{A}$ and consider the function $\phi(t) = x(t) - z(t)$, $t \in [a,b]$. Then, it is evident that

$$f\Big(t, y(t), x'(t), x''(t)\Big) - f\Big(t, y(t), z'(t), z''(t)\Big) = 0, \quad t \in [a,b].$$

Next, we construct the equality

$$f\Big(t,y(t),x'(t),x''(t)\Big)-f\Big(t,y(t),z'(t),x''(t)\Big)+f\Big(t,y(t),z'(t),x''(t)\Big)-f\Big(t,y(t),z'(t),z''(t)\Big)=0,$$

which can be rewritten in the form

$$I_1(t)\phi'(t)+\ I_2(t)\phi''(t)=0,$$

where:

$$\begin{aligned} I_1(t) &= \int_0^1 f_p\Big(t,y(t),z'(t)+\theta(x'(t)-z'(t)),x''(t)\Big)d\theta, \\ I_2(t) &= \int_0^1 f_q\Big(t,y(t),z'(t),z''(t)+\theta(x''(t)-z''(t))\Big)d\theta. \end{aligned}$$

Hence, it follows that the function $\phi(t)$ is a solution to the BVP

$$\phi''(t)+\ \frac{I_1(t)}{I_2(t)}\phi'(t)=0,\quad t\in[a,b],$$

$$\phi(a)=0,\quad \phi'(b)=0.$$

Moreover, it is easy to conclude that $\phi(t)\equiv 0,\ t\in[a,b]$, is the unique solution of the above BVP. Consequently, $x(t)\equiv z(t),\ t\in[a,b]$. The proof of the lemma is complete. □

**Lemma 4.10.** *Let the hypotheses* **H1 - H4** *be hold. If* $y_1(t),y_2(t)\in V_1$ *are such that* $y_1(t)\le y_2(t)$ *for* $t\in[a,b]$, *then*

$$x_1(t)\le x_2(t)\ \ for\ \ t\in[a,b],$$

*where* $x_i=\mathcal{A}y_i, i=1,2.$

*Proof.* Observe that, by Lemma 4.3, we have

$$F\le x_1'(t)\le L,\ \ t\in[a,b],$$

and, by Lemma4.4, it holds

$$m_3\le x_1{}''(t)\ \le M_3,\ \ t\in[a,b].$$

Moreover,

$$Ft\le y_1(t)\le y_2(t)\le Lt,\ \ t\in[a,b].$$

Thus, from

$$f_x(t,x,p,q)\ge 0\quad \text{for}\quad (t,x,p,q)\in T\times[F,L]\times[m_3,M_3]$$

it follows that

$$0=f\Big(t,y_1(t),x_1'(t),x_1''(t)\Big)\le f\Big(t,y_2(t),x_1'(t),x_1''(t)\Big),\ \ t\in[a,b].$$

Hence, for $t\in[a,b]$ we have

$$f\Big(t,y_2(t),x_2'(t),x_2''(t)\Big)-f\Big(t,y_2(t),x_1'(t),x_1''(t)\Big)\le 0$$

and then, as in Lemma 4.9, we construct the inequality

$$f\Big(t, y_2(t), x_1'(t), x_1''(t)\Big) - f\Big(t, y_2(t), x_2'(t), x_1''(t)\Big) + f\Big(t, y_2(t), x_2'(t), x_1''(t)\Big) - f\Big(t, y_2(t), x_2'(t), x_2''(t)\Big) \geq 0$$

from which for $\phi(t) = x_1(t) - x_2(t)$, $t \in [a, b]$, we find

$$\phi''(t) + \frac{J_1(t)}{J_2(t)}\phi'(t) \geq 0, \quad t \in [a, b],$$

where:

$$J_1(t) = \int_0^1 f_p\Big(t, y_2(t), x_2'(t) + \theta(x_1'(t) - x_2'(t)), x_1''(t)\Big) d\theta,$$

$$J_2(t) = \int_0^1 f_q\Big(t, y_2(t), x_2'(t), x_2''(t) + \theta(x_1''(t) - x_2''(t))\Big) d\theta.$$

Furthermore, $\phi(a) = 0$, $\phi'(b) = 0$. Finally, applying Lemma 4.8, we see that

$$\phi(t) \leq 0 \quad \text{for} \quad t \in [a, b],$$

The proof of the lemma is complete. □

**5. Proof of Theorem 3.1.** Consider the sequences $\{u_n\}$ and $\{v_n\}, n = 0, 1, ...$, introduced by the formulas

$$u_{n+1} = \mathcal{A}u_n \quad \text{and} \quad v_{n+1} = \mathcal{A}v_n, \quad n = 0, 1, ...$$

In view of Lemma 4.5, from Lemma 4.3 it follows that

$$Ft = u_0 \leq u_1 \text{ and } v_1 \leq v_0 = Lt.$$

Moreover, Lemma 4.10 and induction arguments imply that

$$u_{n-1} \leq u_n, \quad v_n \leq v_{n-1}, \quad n = 1, 2, ...$$

On the other hand, since

$$u_0 \leq v_0,$$

by Lemma 4.10 and induction arguments, we conclude that

$$u_n \leq v_n, \quad n = 0, 1, ...$$

From the above observation it follows that

$$u_0 \leq u_n \leq v_0, \quad n = 0, 1, ....$$

Therefore, $\{u_n\}$ is uniformly bounded. Furthermore, since, by Lemma 4.3, $\{u_n'\}$ is uniformly bounded, we see that $\{u_n\}$ is equicontinuous. Finally, since, by Lemma 4.4, $\{u_n'''\}$ is uniformly bounded, it follows that the sequence $\{u_n''\}$ is uniformly bounded and equicontinuous. Thus, we can apply the Arzela-Ascoli theorem to conclude that there are a subsequence $\{u_{n_i}\}$ and a function

$u \in C^2[a,b]$ such that $\{u_{n_i}\}$, $\{u'_{n_i}\}$ and $\{u''_{n_i}\}$ are uniformly convergent on $[a,b]$ to $u, u'$ and $u''$ respectively. Now, using the fact that $u_{n_i} = \mathcal{A}u_{n_{i-1}}$ can be rewritten equivalently in the form

$$u_{n_i}(t) = \frac{1}{K}\int_a^t \left( \int_b^r \Big( Ku''_{n_i}(s) + f(s, u_{n_{i-1}}(s), u'_{n_i}(s), u''_{n_i}(s)) \Big) ds \right) dr + B(t-a) + A,$$

letting $i \to +\infty$, we obtain

$$u(t) = \frac{1}{K}\int_a^t \left( \int_b^r \Big( Ku''(s) + f(s, u(s), u'(s), u''(s)) \Big) ds \right) dr + B(t-a) + A,$$

from which it follows that $u(t)$ is a solution to the BVP (1.1).

Remark that, if $x(t)$ is any solution of (1.1), then, by Lemma 4.3, we have

$$u_0(t) \le x(t), \quad t \in [a,b].$$

Applying Lemma 4.10 (it is possible, because $x = \mathcal{A}x$), by induction we obtain

$$u_n(t) \le x(t), \; t \in [a,b], \; n = 0, 1, ...,$$

and then

$$u(t) \le x(t), \quad t \in [a,b],$$

which holds for each solution $x(t) \in C^2[a,b]$ of the problem (1.1). Consequently, it follows that

$$u(t) \equiv u^m(t), \; t \in [a,b].$$

By similar arguments, we conclude that

$$\lim v_n = v^M(t), \quad t \in [a,b].$$

Thus, the proof of the theorem is complete. □

**Aknowledgement.** P.S. Kelevedjiev would like to thank the Ministry of National Economy of Helenic Republic for providing the NATO Science Fellowship (Ref.No. DOO 850/02) and the University of Ioannina for its hospitality.

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